

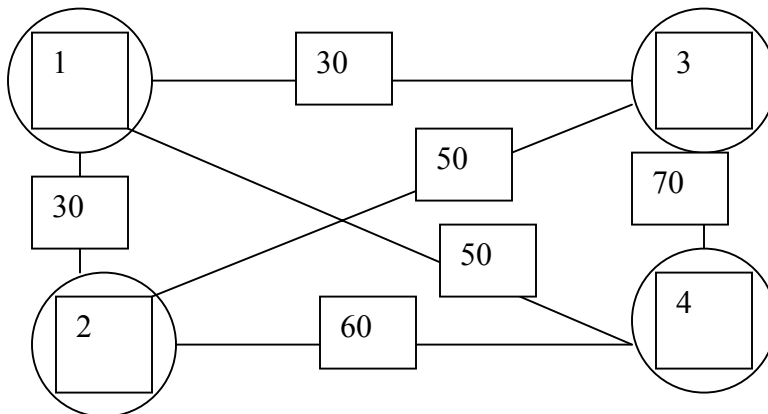
ALL PAIRS SHORTEST PATH DYNAMIC PROGRAMMING SOLUTION



**TIME
COMPLEXITY**
 $T(n) = O(n^3)$

SAMPLE GRAPH

$COST(k, l) = 10(k + l), 1 \leq k, l \leq 4$



In a graph the paths from node i to node j passing at most through nodes as high as k are ${}^k P_{ij} = {}^{k-1} P_{ij} + {}^{k-1} P_{ik} ({}^{k-1} P_{kj})$

This gives a method to obtain the minimum costs between nodes i and j,
 $\min({}^k C_{ij}) = \min({}^{k-1} C_{ij}, ({}^{k-1} C_{ik} + {}^{k-1} C_{kj}))$

STEP 1

$\min({}^k C_{ij})$ for $k=0$ and $i \leq i, j \leq 4$

	Node 1	Node 2	Node 3	Node 4
Node 1	0	30	40	50
Node 2	30	0	50	60
Node 3	40	50	0	70
Node 4	50	60	70	0

STEP 2

$\min({}^k C_{ij})$ for $k=1$ and $i \leq i, j \leq 4$ is $\min({}^{k-1} C_{ij}, ({}^{k-1} C_{ik} + {}^{k-1} C_{kj}))$

$\min({}^1 C_{ij})$ $i=1, j=2$ is $\min({}^0 C_{12}, ({}^0 C_{11} + {}^0 C_{12})) = 30$ //no change

$\min({}^1 C_{ij})$ $i=1, j=3$ is $\min({}^0 C_{13}, ({}^0 C_{11} + {}^0 C_{13})) = 40$ //no change

$\min({}^1 C_{ij})$ $i=1, j=4$ is $\min({}^0 C_{14}, ({}^0 C_{11} + {}^0 C_{14})) = 50$ //no change

$\min({}^1 C_{ij})$ $i=2, j=1$ is $\min({}^0 C_{21}, ({}^0 C_{21} + {}^0 C_{11})) = 30$ //no change

$\min({}^1 C_{ij})$ $i=2, j=3$ is $\min({}^0 C_{23}, ({}^0 C_{21} + {}^0 C_{13})) = \min(50, (30 + 40))=50$

$\min({}^1 C_{ij})$ $i=2, j=4$ is $\min({}^0 C_{24}, ({}^0 C_{21} + {}^0 C_{14})) = \min(60, (30 + 50))=60$

$\min({}^1 C_{ij})$ $i=3, j=1$ is $\min({}^0 C_{31}, ({}^0 C_{31} + {}^0 C_{11})) = 40$ //no change

$\min({}^1 C_{ij})$ $i=3, j=2$ is $\min({}^0 C_{32}, ({}^0 C_{31} + {}^0 C_{12})) = \min(50, (40 + 30))=50$

$\min({}^1 C_{ij})$ $i=3, j=4$ is $\min({}^0 C_{34}, ({}^0 C_{31} + {}^0 C_{14})) = \min(70, (40 + 50))=70$

$\min({}^1 C_{ij})$ $i=4, j=1$ is $\min({}^0 C_{41}, ({}^0 C_{41} + {}^0 C_{14})) = \min(50, (50 + 50))=50$

$\min({}^1 C_{ij})$ $i=4, j=2$ is $\min({}^0 C_{42}, ({}^0 C_{41} + {}^0 C_{12})) = \min(60, (50 + 30))=60$

$\min({}^1 C_{ij})$ $i=4, j=3$ is $\min({}^0 C_{43}, ({}^0 C_{41} + {}^0 C_{13})) = \min(70, (50 + 40))=70$

THE SHORTEST ALL PATHS PASSING THROUGH NODES AS HIGH AS 1

	Node 1	Node 2	Node 3	Node 4
Node 1	0	30	40	50
Node 2	30	0	50	60
Node 3	40	50	0	70
Node 4	50	60	70	0

STEP 3

$\min({}^k C_{ij})$ for $k=2$ and $i \leq i, j \leq 4$ is $\min({}^{k-1} C_{ij}, ({}^{k-1} C_{ik} + {}^{k-1} C_{kj}))$

$\min({}^2 C_{ij})$ $i=1, j=2$ is $\min({}^1 C_{12}, ({}^1 C_{12} + {}^1 C_{22})) = 30$ //no change
 $\min({}^2 C_{ij})$ $i=1, j=3$ is $\min({}^1 C_{13}, ({}^1 C_{12} + {}^1 C_{23})) = \min(40, (30 + 50))=40$
 $\min({}^2 C_{ij})$ $i=1, j=4$ is $\min({}^1 C_{14}, ({}^1 C_{12} + {}^1 C_{24})) = \min(50, (30 + 60))=50$

$\min({}^2 C_{ij})$ $i=2, j=1$ is $\min({}^1 C_{21}, ({}^1 C_{22} + {}^1 C_{21})) = 30$ //no change
 $\min({}^2 C_{ij})$ $i=2, j=3$ is $\min({}^1 C_{23}, ({}^1 C_{22} + {}^1 C_{23})) = \min(50, (40 + 50))=50$
 $\min({}^2 C_{ij})$ $i=2, j=4$ is $\min({}^1 C_{24}, ({}^1 C_{22} + {}^1 C_{24})) = \min(60, (40 + 60))=60$

$\min({}^2 C_{ij})$ $i=3, j=1$ is $\min({}^1 C_{31}, ({}^1 C_{32} + {}^1 C_{21})) = (40, (50 + 30)) = 40$
 $\min({}^2 C_{ij})$ $i=3, j=2$ is $\min({}^1 C_{32}, ({}^1 C_{32} + {}^1 C_{22})) = 50$
 $\min({}^2 C_{ij})$ $i=3, j=4$ is $\min({}^1 C_{34}, ({}^1 C_{32} + {}^1 C_{24})) = \min(70, (50 + 60))=70$

$\min({}^2 C_{ij})$ $i=4, j=1$ is $\min({}^1 C_{41}, ({}^1 C_{42} + {}^1 C_{24})) = \min(50, (60 + 60))=50$
 $\min({}^2 C_{ij})$ $i=4, j=2$ is $\min({}^1 C_{42}, ({}^1 C_{42} + {}^1 C_{22})) = \min(60, (60 + 40))=60$
 $\min({}^2 C_{ij})$ $i=4, j=3$ is $\min({}^1 C_{43}, ({}^1 C_{42} + {}^1 C_{23})) = \min(70, (60 + 50))=70$

THE SHORTEST ALL PATHS PASSING THROUGH NODES AS HIGH AS 2

	Node 1	Node 2	Node 3	Node 4
Node 1	0	30	40	50
Node 2	30	0	50	60
Node 3	40	50	0	70
Node 4	50	60	70	0

STEP 4

$\min({}^k C_{ij})$ for $k=3$ and $i \leq i, j \leq 4$ is $\min({}^{k-1} C_{ij}, ({}^{k-1} C_{ik} + {}^{k-1} C_{kj}))$

$\min({}^3 C_{ij})$ $i=1, j=2$ is $\min({}^2 C_{12}, ({}^2 C_{13} + {}^2 C_{32})) = \min(30, (40 + 50))$
 $\min({}^3 C_{ij})$ $i=1, j=3$ is $\min({}^2 C_{13}, ({}^2 C_{13} + {}^2 C_{33})) = 40$ //no change
 $\min({}^3 C_{ij})$ $i=1, j=4$ is $\min({}^2 C_{14}, ({}^2 C_{13} + {}^2 C_{34})) = \min(50, (4 + 70))$

$\min({}^3 C_{ij})$ $i=2, j=1$ is $\min({}^2 C_{21}, ({}^2 C_{23} + {}^2 C_{31})) = (30, (50 + 40))=30$
 $\min({}^3 C_{ij})$ $i=2, j=3$ is $\min({}^2 C_{23}, ({}^2 C_{23} + {}^2 C_{33})) = 50$
 $\min({}^3 C_{ij})$ $i=2, j=4$ is $\min({}^2 C_{24}, ({}^2 C_{23} + {}^2 C_{34})) = \min(60, (50 + 70))=60$

$\min({}^3 C_{ij})$ $i=3, j=1$ is $\min({}^2 C_{31}, ({}^2 C_{33} + {}^2 C_{31})) = 40$ //no change
 $\min({}^3 C_{ij})$ $i=3, j=2$ is $\min({}^2 C_{32}, ({}^2 C_{33} + {}^2 C_{32})) = 50$
 $\min({}^3 C_{ij})$ $i=3, j=4$ is $\min({}^2 C_{34}, ({}^2 C_{33} + {}^2 C_{34})) = 70$

$\min({}^3 C_{ij})$ $i=4, j=1$ is $\min({}^2 C_{41}, ({}^2 C_{43} + {}^2 C_{34})) = \min(50, (70 + 70))=50$

$$\min({}^3C_{ij} \mid i=4, j=2) \text{ is } \min({}^2C_{42}, ({}^2C_{43} + {}^2C_{32})) = \min(60, (70 + 50)) = 60$$

$$\min({}^3C_{ij} \mid i=4, j=3) \text{ is } \min({}^2C_{43}, ({}^2C_{43} + {}^2C_{33})) = 70$$

THE SHORTEST ALL PATHS PASSING THROUGH NODES AS HIGH AS 3

	Node 1	Node 2	Node 3	Node 4
Node 1	0	30	40	50
Node 2	30	0	50	60
Node 3	40	50	0	70
Node 4	50	60	70	0

STEP 5

$$\min({}^kC_{ij} \text{ for } k=4 \text{ and } i \leq i, j \leq 4) \text{ is } \min({}^{k-1}C_{ij}, ({}^{k-1}C_{ik} + {}^{k-1}C_{kj}))$$

$$\min({}^4C_{ij} \mid i=1, j=2) \text{ is } \min({}^3C_{12}, ({}^3C_{14} + {}^3C_{42})) = \min(30, (50 + 60)) = 30$$

$$\min({}^4C_{ij} \mid i=1, j=3) \text{ is } \min({}^3C_{13}, ({}^3C_{14} + {}^3C_{43})) = \min(40, (50 + 70)) = 40$$

$$\min({}^4C_{ij} \mid i=1, j=4) \text{ is } \min({}^3C_{14}, ({}^3C_{14} + {}^3C_{44})) = 50$$

$$\min({}^4C_{ij} \mid i=2, j=1) \text{ is } \min({}^3C_{21}, ({}^3C_{24} + {}^3C_{41})) = (30, (60 + 50)) = 30$$

$$\min({}^4C_{ij} \mid i=2, j=3) \text{ is } \min({}^3C_{23}, ({}^3C_{24} + {}^3C_{43})) = \min(50, (60 + 70)) = 50$$

$$\min({}^4C_{ij} \mid i=2, j=4) \text{ is } \min({}^3C_{24}, ({}^3C_{24} + {}^3C_{44})) = 60$$

$$\min({}^4C_{ij} \mid i=3, j=1) \text{ is } \min({}^3C_{31}, ({}^3C_{34} + {}^3C_{41})) = \min(40, (70 + 50)) = 40$$

$$\min({}^4C_{ij} \mid i=3, j=2) \text{ is } \min({}^3C_{32}, ({}^3C_{34} + {}^3C_{42})) = \min(50, (70 + 60)) = 50$$

$$\min({}^4C_{ij} \mid i=3, j=4) \text{ is } \min({}^3C_{34}, ({}^3C_{34} + {}^3C_{44})) = 70$$

$$\min({}^4C_{ij} \mid i=4, j=1) \text{ is } \min({}^3C_{41}, ({}^3C_{44} + {}^3C_{44})) = 50$$

$$\min({}^4C_{ij} \mid i=4, j=2) \text{ is } \min({}^3C_{42}, ({}^3C_{44} + {}^3C_{42})) = 60$$

$$\min({}^4C_{ij} \mid i=4, j=3) \text{ is } \min({}^3C_{43}, ({}^3C_{44} + {}^3C_{43})) = 70$$

THE SHORTEST ALL PATHS PASSING THROUGH NODES AS HIGH AS 4

	Node 1	Node 2	Node 3	Node 4
Node 1	0	30	40	50
Node 2	30	0	50	60
Node 3	40	50	0	70
Node 4	50	60	70	0