

# OPTIMAL BINARY SEARCH TREES



**COMPLEXITY**  
 $T(n) = O(n^3)$

**COMPLEXITY**  
**USING KNUTH'S OPTIMISATION**  
 $T(n) = O(n^2)$

List = {eyeore, kanga, piglet, pooh, robin, roo, tigger}

= {a1, a2, a3, a4, a5, a6, a7}

for j = 1 to 7 : P(j) = 1

for j = 0 to 7: Q(j) = 1

Assume the probabilities are multiplied by 15.

RECURRENCES:

$$W(I,J) = P(J) + Q(J) + W(I,J-1)$$

$$C(I,J) = W(I,J) + \min[\text{for } K = I+1 \text{ to } J] \{ \text{COST}(I, K-1) + \text{COST}(K,J) \}$$

R(I,J) is the K which minimizes C(I,J)

For the data given

$$W(I,J) = P(J) + Q(J) + W(I,J-1) = 2 + W(I,J-1)$$

## STEP I



$$C(I,I)=0 \text{ i.e. } C(0,0) = C(1,1) = C(2,2) = C(3,3) = C(4,4)$$

$$=C(5,5)=C(6,6)=C(7,7)=0$$

$$W(I,I)=0 \text{ i.e. } W(0,0) = W(1,1) = W(2,2) = W(3,3) = W(4,4)$$

$$=W(5,5)=W(6,6)=W(7,7)=1$$

$$R(I,I)=0 \text{ i.e. } R(0,0) = R(1,1) = R(2,2) = R(3,3) = R(4,4)$$

$$=R(5,5)=R(6,6)=R(7,7)=0$$

	I=0	I=1	I=2	I=3	I=4	I=5	I=6	I=7
L=0	C(0,0)=0	C(1,1)=0	C(2,2)=0	C(3,3)=3	C(4,4)=0	C(5,5)=0	C(6,6)=0	C(7,7)=0
Ti,i	W(0,0)=1	W(1,1)=1	W(2,2)=1	W(3,3)=1	W(4,4)=0	W(5,5)=0	W(6,6)=0	W(7,7)=0
	R(0,0)=0	R(1,1)=0	R(2,2)=0	R(3,3)=0	R(4,4)=0	R(5,5)=0	R(6,6)=0	R(7,7)=0

## STEP II

$$T_{i,i+1}$$

$$W(I,I+1) = P(I+1) + Q(I+1) + W(I,I) \\ = 3$$

$$\text{i.e. } W(0,1)=W(1,2)=W(2,3)=W(3,4)=W(4,5)=W(5,6)=W(6,7)=3$$

$$C(I,I+1)=W(I,I+1) + \min[\text{for } K = I+1 \text{ to } I+1] \{ \text{COST}(I,K-1) + \\ \text{COST}(K,I+1) \} \\ = 3 + [\text{COST}(I,I) + \text{COST}(I+1,I+1)] \\ = 3$$

$$C(0,1)=C(1,2)=C(2,3)=C(3,4)=C(4,5)=C(5,6)=C(6,7)=3$$

$$R(I,I+1)=I+1$$

$$R(0,1)=1, R(1,2)=2, R(2,3)=3, R(3,4)=4, R(4,5)=5, R(5,6)=6, R(6,7)=7$$

	I=0	I=1	I=2	I=3	I=4	I=5	I=6	I=7
L=0	C(0,0)=0	C(1,1)=0	C(2,2)=0	C(3,3)=3	C(4,4)=0	C(5,5)=0	C(6,6)=0	C(7,7)=0
T <sub>i,i</sub>	W(0,0)=1	W(1,1)=1	W(2,2)=1	W(3,3)=1	W(4,4)=0	W(5,5)=0	W(6,6)=0	W(7,7)=0
	R(0,0)=0	R(1,1)=0	R(2,2)=0	R(3,3)=0	R(4,4)=0	R(5,5)=0	R(6,6)=0	R(7,7)=0
L=1	C(0,1)=3	C(1,2)=3	C(2,3)=3	C(3,4)=3	C(4,5)=3	C(5,6)=3	C(6,7)=3	
	W(0,1)=3	W(1,2)=3	W(2,3)=3	W(3,4)=3	W(4,5)=3	W(5,6)=3	W(6,7)=3	
	R(0,1)=1	R(1,2)=2	R(2,3)=3	R(3,4)=4	R(4,5)=5	R(5,6)=6	R(6,7)=7	

## STEP III

$T_{i,i+2}$

$$\begin{aligned}W(I,I+2) &= P(I+2) + Q(I+2) + W(I,I+1) \\ &= 5\end{aligned}$$

i.e.  $W(0,2)=5, W(1,3)=5, W(2,4)=5, W(3,5)=5, W(4,6)=5, W(5,7)=5, W(6,8)=5$

$$\begin{aligned}C(I,I+2) &= W(I,I+2) + \min[\text{for } k= I+1, I+2] \{ \text{COST}(I,K-1) + \text{COST}(K,I+2) \} \\ &= 5 + \min \{ [\text{COST}(I,I) + \text{COST}(I+1,I+2)], \text{COST}(I, I+1) + \\ &\text{COST}(I+2,I+2) \} \\ &= 5 + \min([0+3], [3+0]) \\ &= 8\end{aligned}$$

$$R(I,I+2) = I+1 \text{ OR } I+2$$

i.e.  $R(0,2)=1 \text{ OR } 2, R(1,3)=2 \text{ OR } 3, R(2,4)=3 \text{ OR } 4, R(3,5)=4 \text{ OR } 5, R(4,6)=5$   
 $\text{OR } 6, R(5,7)=6 \text{ OR } 7, R(6,8)=7 \text{ OR } 8$

### KNUTH'S OPTIMISATION

$$C(I,I+2) = W(I,I+2) + \min_{R(I,I+1) < K \leq R(I+1,I+2)} \{ \text{COST}(I,K-1) + \text{COST}(K,I+2) \}$$

$$\begin{aligned}&[\text{sample:for } I = 0] \\ &= W(0,2) + \min_{R(0,1) < K \leq R(1,2)} \{ \text{COST}(I,K-1) + \text{COST}(K,I+2) \} [\text{for } I = 0] \\ &= W(0,2) + \min_{1 < K=2} \{ \text{COST}(I,K-1) + \text{COST}(K,I+2) \} [\text{for } I = 0] \\ &= W(0,2) + \min_{K=2} \{ \text{COST}(I,K-1) + \text{COST}(K,I+2) \} [\text{for } I = 0] \\ &= W(0,2) + \text{COST}(0,1) + \text{COST}(2,2) \\ &= 8\end{aligned}$$

	I=0	I=1	I=2	I=3	I=4	I=5	I=6	I=7
L=0	C(0,0)=0	C(1,1)=0	C(2,2)=0	C(3,3)=3	C(4,4)=0	C(5,5)=0	C(6,6)=0	C(7,7)=0
Ti,i	W(0,0)=1	W(1,1)=1	W(2,2)=1	W(3,3)=1	W(4,4)=0	W(5,5)=0	W(6,6)=0	W(7,7)=0
	R(0,0)=0	R(1,1)=0	R(2,2)=0	R(3,3)=0	R(4,4)=0	R(5,5)=0	R(6,6)=0	R(7,7)=0
L=1	C(0,1)=3	C(1,2)=3	C(2,3)=3	C(3,4)=3	C(4,5)=3	C(5,6)=3	C(6,7)=3	
	W(0,1)=3	W(1,2)=3	W(2,3)=3	W(3,4)=3	W(4,5)=3	W(5,6)=3	W(6,7)=3	
	R(0,1)=1	R(1,2)=2	R(2,3)=3	R(3,4)=4	R(4,5)=5	R(5,6)=6	R(6,7)=7	
L=2	C(0,2)=8	C(1,3)=8	C(2,4)=8	C(3,5)=8	C(4,6)=8	C(5,7)=8		
	W(0,2)=5	W(1,3)=5	W(2,4)=5	W(3,5)=5	W(4,6)=5	W(5,7)=5		
	R(0,2) = 1 or 2	R(1,3) = 2 or 3	R(2,4) = 3 or 4	R(3,5) = 4 or 5	R(4,6) = 5 or 6	R(5,7) = 6 or 7		

## STEP IV

$$T_{i,i+3}$$

$$W(I,I+3)=P(I+3) + Q(I+3) + W(I,I+2) =7$$

$$C(I,I+3)=7 + \min[\text{for } K = I+1, I+2, I+3] \{ [COST(I, K-1) + COST(K, I+3)] \}$$

$$=7 + \min \{ [COST(I, I) + COST(I+1, I+3)], [COST(I, I+1) +$$

$$COST(I+2, I+3)], [COST(I, I+2) + COST(I+3, I+3)] \}$$

$$= 7 + \min \{ [0 + 8], [3 + 3], [8 + 0] \}$$

$$= 13$$

$$R(I, I+3) = I+2$$

$$\text{i.e. } R(0,3)=2, R(1,4)=3, R(2,5)=4, R(3,6)=5, R(4,7)=6, R(5,8)=7$$

### KNUTH'S OPTIMISATION

$$C(I,I+3) = W(I,I+3) + \min[R(I,I+1) < K \leq R(I+1,I+3)] \\ \{[COST(I,K-1) + COST(K,I+2)]\}$$

$$\begin{aligned} & \text{[sample:for } I = 0] \\ & = W(0,3) + \min[R(0,1) < K \leq R(1,3)] \\ & \{[COST(I,K-1) + COST(K,I+3)] \text{ [for } I = 0] \\ & = W(0,3) + \min[1 < K = 2] \\ & \{[COST(I,K-1) + COST(K,I+3)] \text{ [for } I = 0] \\ & = W(0,3) + \min[K = 2] \\ & \{[COST(I,K-1) + COST(K,I+3)] \text{ [for } I = 0] \\ & = W(0,3) + COST(0,1) + COST(2,3) \\ & = 13 \end{aligned}$$

L=0	C(0,0)=0	C(1,1)=0	C(2,2)=0	C(3,3)=3	C(4,4)=0	C(5,5)=0	C(6,6)=0	C(7,7)=0
Ti,i	W(0,0)=1	W(1,1)=1	W(2,2)=1	W(3,3)=1	W(4,4)=0	W(5,5)=0	W(6,6)=0	W(7,7)=0
	R(0,0)=0	R(1,1)=0	R(2,2)=0	R(3,3)=0	R(4,4)=0	R(5,5)=0	R(6,6)=0	R(7,7)=0
L=1	C(0,1)=3	C(1,2)=3	C(2,3)=3	C(3,4)=3	C(4,5)=3	C(5,6)=3	C(6,7)=3	
	W(0,1)=3	W(1,2)=3	W(2,3)=3	W(3,4)=3	W(4,5)=3	W(5,6)=3	W(6,7)=3	
	R(0,1)=1	R(1,2)=2	R(2,3)=3	R(3,4)=4	R(4,5)=5	R(5,6)=6	R(6,7)=7	
L=2	C(0,2)=8	C(1,3)=8	C(2,4)=8	C(3,5)=8	C(4,6)=8	C(5,7)=8		
	W(0,2)=5	W(1,3)=5	W(2,4)=5	W(3,5)=5	W(4,6)=5	W(5,7)=5		
	R(0,2) = 1 or 2	R(1,3) = 2 or 3	R(2,4) = 3 or 4	R(3,5) = 4 or 5	R(4,6) = 5 or 6	R(5,7) = 6 or 7		
L=3	C(0,3)=13	C(1,4)=13	C(2,5)=13	C(3,6)=15	C(4,7)=13			
	W(0,3)=7	W(1,4)=7	W(2,5)=7	W(3,6)=7	W(4,7)=7			
	R(0,3) = 2	R(1,4) = 3	R(2,5) = 4	R(3,6) = 5	R(4,7) = 6			

L=4	C(0,4)=20 W(0,4)=9 R(0,4) = 2 or 3	C(1,5)=20 W(1,5)=9 R(1,5) = 3 or 4	C(2,6)=20 W(2,6)=9 R(2,6) = 4 or 5	C(3,7)=20 W(3,7)=9 R(3,7) = 5 or 6				
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## STEP V

# T<sub>i,i+4</sub>

$$W(I,I+4)=P(I+4) + Q(I+4) + W(I,I+3)= 9$$

$$C(I,I+4)=9 + \min[\text{for } K = I+1,I+2,I+3,I+4] \{[COST(I,K-1) + COST(K,I+4)]\}$$

$$=9 + \min\{[COST(I,I) + COST(I+1,I+4)], [COST(I,I+1) +$$

$$COST(I+2,I+4)],$$

$$[COST(I,I+2) + COST(I+3,I+4)], [COST(I,I+3) +$$

$$COST(I+4,I+4)]\}$$

$$= 9 + \min\{[0 + 15], [3 + 8],[8 + 3],[15+0]\}$$

$$= 20$$

$$R(I,I+4)=I+2 \text{ OR } I+3$$

$$\text{i.e. } R(0,4)=2 \text{ OR } 3, R(1,5)=3 \text{ OR } 4, R(2,6)= 4 \text{ OR } 5, R(3,7)=5 \text{ OR } 6, R(4,8)=6$$

$$\text{OR } 7$$

### KNUTH'S OPTIMISATION

$$C(I,I+4) = W(I,I+4) + \min_{R(I,I+1) < K \leq R(I+1,I+4)} \{ \text{COST}(I,K-1) + \text{COST}(K,I+4) \}$$

$$\begin{aligned} & \text{[sample:for } I = 0] \\ & = W(0,4) + \min_{R(0,1) < K \leq R(1,4)} \{ \text{COST}(I,K-1) + \text{COST}(K,I+4) \} \text{[for } I = 0] \\ & = W(0,4) + \min_{3 < K = 3} \{ \text{COST}(I,K-1) + \text{COST}(K,I+4) \} \text{[for } I = 0] \\ & = W(0,4) + \min_{K=3} \{ \text{COST}(I,K-1) + \text{COST}(K,I+4) \} \text{[for } I = 0] \\ & = W(0,4) + \text{COST}(0,2) + \text{COST}(3,3) \\ & = 20 \end{aligned}$$

## STEP VI

$T_{i,i+5}$

$$W(I,I+5) = P(I+5) + Q(I+5) + W(I,I+4) = 11$$

$$C(I,I+5) = W(I,I+5) + \min_{\text{for } K = I+1, I+2, I+3, I+4, I+5} \{ \text{COST}(I,K-1) + \text{COST}(K,I+5) \}$$

$$= 11 + \min \{ [\text{COST}(I, I) + \text{COST}(I+1, I+5)], [\text{COST}(I, I+1) + \text{COST}(I+2, I+5)],$$

$$[\text{COST}(I, I+2) + \text{COST}(I+3, I+5)], [\text{COST}(I, I+3) + \text{COST}(I+4, I+5)],$$

$$[\text{COST}(I, I+4) + \text{COST}(I+5, I+5)] \}$$

$$= 11 + \min \{ [0 + 20], [3 + 13], [8 + 8], [13 + 3], [20 + 0] \}$$

$$= 11 + 16$$

$$= 27$$

$$\text{ROOT}(I,I+5)=I+3$$

$$R(0,5)=3,R(1,6)=4,R(2,7)=5$$

**KNUTH'S OPTIMISATION**  

$$C(I,I+5) = W(I,I+5) + \min[R(I,I+1) < K \leq R(I+1,I+5)]$$

$$\{ [COST(I,K-1) + COST(K,I+5)]$$

**[sample:for I = 0]**  

$$= W(0,5) + \min[R(0,4) < K \leq R(1,5)]$$

$$\{ [COST(I,K-1) + COST(K,I+5)] [for I = 0]$$

$$= W(0,5) + \min[1 < K = 3]$$

$$\{ [COST(I,K-1) + COST(K,I+5)] [for I = 0]$$

$$= W(0,5) + \min[K = 3]$$

$$\{ [COST(0,2) + COST(3,5)] \}$$

$$= 11 + \min\{ [8+8] \}$$

$$= 27$$

	I=0	I=1	I=2	I=3	I=4	I=5	I=6	I=7
L=0	C(0,0)=0	C(1,1)=0	C(2,2)=0	C(3,3)=3	C(4,4)=0	C(5,5)=0	C(6,6)=0	C(7,7)=0
Ti,i	W(0,0)=1	W(1,1)=1	W(2,2)=1	W(3,3)=1	W(4,4)=0	W(5,5)=0	W(6,6)=0	W(7,7)=0
	R(0,0)=0	R(1,1)=0	R(2,2)=0	R(3,3)=0	R(4,4)=0	R(5,5)=0	R(6,6)=0	R(7,7)=0
L=1	C(0,1)=3	C(1,2)=3	C(2,3)=3	C(3,4)=3	C(4,5)=3	C(5,6)=3	C(6,7)=3	
	W(0,1)=3	W(1,2)=3	W(2,3)=3	W(3,4)=3	W(4,5)=3	W(5,6)=3	W(6,7)=3	
	R(0,1)=1	R(1,2)=2	R(2,3)=3	R(3,4)=4	R(4,5)=5	R(5,6)=6	R(6,7)=7	
L=2	C(0,2)=8	C(1,3)=8	C(2,4)=8	C(3,5)=8	C(4,6)=8	C(5,7)=8		
	W(0,2)=5	W(1,3)=5	W(2,4)=5	W(3,5)=5	W(4,6)=5	W(5,7)=5		
	R(0,2) = 1	R(1,3) = 2	R(2,4) = 3	R(3,5) = 4	R(4,6) = 5	R(5,7) =		

	or 2	or 3	or 4	or 5	or 6	6 or 7		
L=3	C(0,3)=13 W(0,3)=7 R(0,3) =2	C(1,4)=13 W(1,4)=7 R(1,4) =3	C(2,5)=13 W(2,5)=7 R(2,5) =4	C(3,6)=13 W(3,6)=7 R(3,6) =5	C(4,7)=13 W(4,7)=7 R(4,7) =6			
L=4	C(0,4)=20 W(0,4)=9 R(0,4) = 2 or 3	C(1,5)=20 W(1,5)=9 R(1,5) = 3 or 4	C(2,6)=20 W(2,6)=9 R(2,6) = 4 or 5	C(3,7)=20 W(3,7)=9 R(3,7) = 5 or 6				
L=5	C(0,5)=27 W(0,5)=11 R(0,5) = 3	C(1,6)=27 W(1,6)=11 R(1,6) = 3	C(2,7)=27 W(2,7)=11 R(2,7) = 3					

## STEP VII

$$T_{i,i+6}$$

$$W(I,I+6)=P(I+6) + Q(I+6) + W(I,5)=13$$

$$C(I,I+6)=W(I,I+6) + \min[\text{for } K = I+1, I+2, I+3, I+4, I+5, I+6] \{[\text{COST}(I, K-1) + \text{COST}(I, I+6)]$$

$$= 13 + \min \{[\text{COST}(I, I) + \text{COST}(I+1, I+6)], [\text{COST}(I, I+1) + \text{COST}(I+2, I+6)],$$

$$[\text{COST}(I, I+2) + \text{COST}(I+3, I+6)], [\text{COST}(I, I+3) + \text{COST}(I+4, I+6)],$$

$$[\text{COST}(I, I+4) + \text{COST}(I+5, I+6)], [\text{COST}(I, I+5) + \text{COST}(I+6, I+6)]$$

$$= 13 + \min \{[0 + 25], [3 + 20], [8 + 15], [20 + 3], [25 + 0]\}$$

$$=30$$

$R(I, I+6) = I+2$  OR  $I+3$  OR  $I+4$

$R(0,6) = 2$  OR  $3$  OR  $4$ ,  $R(1,7) = 3$  OR  $4$  OR  $5$

**KNUTH'S OPTIMISATION**

$$C(I, I+6) = W(I, I+6) + \min_{R(I, I+1) < K \leq R(I+1, I+6)} \{COST(I, K-1) + COST(K, I+6)\}$$

[sample: for  $I = 0$ ]

$$= W(0,6) + \min_{R(0,5) < K \leq R(1,6)} \{COST(I, K-1) + COST(K, I+6)\} \text{ [for } I = 0 \text{]}$$

$$= W(0,6) + \min_{1 < K = 2} \{COST(I, K-1) + COST(K, I+6)\} \text{ [for } I = 0 \text{]}$$

$$= W(0,6) + \min_{K = 3} \{COST(I, K-1) + COST(K, I+6)\} \text{ [for } I = 0 \text{]}$$

$$= W(0,6) + COST(0,2) + COST(3,6)$$

$$= 30$$

	I=0	I=1	I=2	I=3	I=4	I=5	I=6	I=7
L=0	C(0,0)=0	C(1,1)=0	C(2,2)=0	C(3,3)=3	C(4,4)=0	C(5,5)=0	C(6,6)=0	C(7,7)=0
Ti,i	W(0,0)=1	W(1,1)=1	W(2,2)=1	W(3,3)=1	W(4,4)=0	W(5,5)=0	W(6,6)=0	W(7,7)=0
	R(0,0)=0	R(1,1)=0	R(2,2)=0	R(3,3)=0	R(4,4)=0	R(5,5)=0	R(6,6)=0	R(7,7)=0
L=1	C(0,1)=3	C(1,2)=3	C(2,3)=3	C(3,4)=3	C(4,5)=3	C(5,6)=3	C(6,7)=3	
	W(0,1)=3	W(1,2)=3	W(2,3)=3	W(3,4)=3	W(4,5)=3	W(5,6)=3	W(6,7)=3	
	R(0,1)=1	R(1,2)=2	R(2,3)=3	R(3,4)=4	R(4,5)=5	R(5,6)=6	R(6,7)=7	
L=2	C(0,2)=8	C(1,3)=8	C(2,4)=8	C(3,5)=8	C(4,6)=8	C(5,7)=8		
	W(0,2)=5	W(1,3)=5	W(2,4)=5	W(3,5)=5	W(4,6)=5	W(5,7)=5		
	R(0,2) = 1 or 2	R(1,3) = 2 or 3	R(2,4) = 3 or 4	R(3,5) = 4 or 5	R(4,6) = 5 or 6	R(5,7) = 6 or 7		
L=3	C(0,3)=13	C(1,4)=13	C(2,5)=13	C(3,6)=13	C(4,7)=13			

	W(0,3)=7 R(0,3)=2	W(1,4)=7 R(1,4)=3	W(2,5)=7 R(2,5)=4	W(3,6)=7 R(3,6)=5	W(4,7)=7 R(4,7)=6			
L=4	C(0,4)=20 W(0,4)=9 R(0,4)=2 or 3	C(1,5)=20 W(1,5)=9 R(1,5)=3 or 4	C(2,6)=20 W(2,6)=9 R(2,6)=4 or 5	C(3,7)=20 W(3,7)=9 R(3,7)=5 or 6				
L=5	C(0,5)=27 W(0,5)=11 R(0,5)=3	C(1,6)=27 W(1,6)=11 R(1,6)=3	C(2,7)=27 W(2,7)=11 R(2,7)=3					
L=6	C(0,6)=30 W(0,6)=13 R(0,6)=2 or 3 or 4	C(1,7)=30 W(1,7)=13 R(1,7)=3 or 4 or 5						

## STEP VIII

$$T_{i,i+7}$$

$$W(I,I+7) = P(I+7) + Q(I+7) + W(I,I+6) = 15$$

$$C(I,I+7) = W(I,I+7) + \min[\text{for } K=I+1, I+2, I+3, I+4, I+5, I+6, I+7] \{ [COST(I, K-1) + COST(K, I+7)] \}$$

$$= 15 + \min \{ [COST(I, I) + COST(I+1, I+7)], [COST(I, I+1) + COST(I+2, I+7)], [COST(I, I+2) + COST(I+3, I+7)], [COST(I, I+3) + COST(I+4, I+7)], [COST(I, I+4) + COST(I+5, I+7)], [COST(I, I+5) + COST(I+6, I+7)], [COST(I, I+6) + COST(I+7, I+7)] \}$$

$$= 15 + \min\{[0 + 30], [3 + 27], [8 + 20], [15 + 15], [20 + 8], [27 + 3], [30 + 0]\}$$

$$= 43$$

$$R(I, I+7) = I+2, I+3, I+5, I+6$$

$$R(0, 7) = 2, 3, 5 \text{ OR } 6$$

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$$C(I, I+7) = W(I, I+7) + \min_{R(I, I+1) < K \leq R(I+1, I+7)} \{ \text{COST}(I, K-1) + \text{COST}(K, I+7) \}$$

[sample: for I = 0]

$$= W(0, 7) + \min_{R(0, 6) < K \leq R(1, 7)} \{ \text{COST}(I, K-1) + \text{COST}(K, I+7) \} \text{ [for I = 0]}$$

$$= W(0, 7) + \min_{3 < K \leq 3} \{ \text{COST}(I, K-1) + \text{COST}(K, I+7) \} \text{ [for I = 0]}$$

$$= W(0, 7) + \min_{K=3} \{ \text{COST}(I, K-1) + \text{COST}(K, I+7) \} \text{ [for I = 0]}$$

$$= W(0, 7) + \text{COST}(0, 2) + \text{COST}(3, 7)$$

$$= 43$$

	I=0	I=1	I=2	I=3	I=4	I=5	I=6	I=7
L=0	C(0,0)=0	C(1,1)=0	C(2,2)=0	C(3,3)=3	C(4,4)=0	C(5,5)=0	C(6,6)=0	C(7,7)=0
Ti,i	W(0,0)=1	W(1,1)=1	W(2,2)=1	W(3,3)=1	W(4,4)=0	W(5,5)=0	W(6,6)=0	W(7,7)=0
	R(0,0)=0	R(1,1)=0	R(2,2)=0	R(3,3)=0	R(4,4)=0	R(5,5)=0	R(6,6)=0	R(7,7)=0
L=1	C(0,1)=3	C(1,2)=3	C(2,3)=3	C(3,4)=3	C(4,5)=3	C(5,6)=3	C(6,7)=3	
	W(0,1)=3	W(1,2)=3	W(2,3)=3	W(3,4)=3	W(4,5)=3	W(5,6)=3	W(6,7)=3	
	R(0,1)=1	R(1,2)=2	R(2,3)=3	R(3,4)=4	R(4,5)=5	R(5,6)=6	R(6,7)=7	
L=2	C(0,2)=8	C(1,3)=8	C(2,4)=8	C(3,5)=8	C(4,6)=8	C(5,7)=8		
	W(0,2)=5	W(1,3)=5	W(2,4)=5	W(3,5)=5	W(4,6)=5	W(5,7)=5		
	R(0,2) = 1 or 2	R(1,3) = 2 or 3	R(2,4) = 3 or 4	R(3,5) = 4 or 5	R(4,6) = 5 or 6	R(5,7) = 6 or 7		

L=3	C(0,3)=13 W(0,3)=7 R(0,3) =2	C(1,4)=13 W(1,4)=7 R(1,4) =3	C(2,5)=13 W(2,5)=7 R(2,5) =4	C(3,6)=13 W(3,6)=7 R(3,6) =5	C(4,7)=13 W(4,7)=7 R(4,7) =6			
L=4	C(0,4)=20 W(0,4)=9 R(0,4) = 2 or 3	C(1,5)=20 W(1,5)=9 R(1,5) = 3 or 4	C(2,6)=20 W(2,6)=9 R(2,6) = 4 or 5	C(3,7)=20 W(3,7)=9 R(3,7) = 5 or 6				
L=5	C(0,5)=27 W(0,5)=11 R(0,5) = 3	C(1,6)=27 W(1,6)=11 R(1,6) = 3	C(2,7)=27 W(2,7)=11 R(2,7) = 3					
L=6	C(0,6)=30 W(0,6)=13 R(0,6) = 2 or 3 or 4	C(1,7)=30 W(1,7)=13 R(1,7) = 3 or 4 or 5						
L=7	C(0,7)=43 W(0,7)=15 R(0,7) = 2 or 3 or 5 or 6							